# ON THE ADM EQUATIONS FOR GENERAL RELATIVITY

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The Arnowitt–Deser–Misner (ADM) evolution equations for the induced metric and the extrinsic-curvature tensor of the spacelike surfaces which foliate the space-time manifold in canonical general relativity are a first-order system of quasi-linear partial differential equations, supplemented by the constraint equations. Such equations are here mapped into another first-order system. In particular, an evolution equation for the trace of the extrinsic-curvature tensor $K$ is obtained whose solution is related to a discrete spectral resolution of a three-dimensional elliptic operator $\mathcal{P}$ of Laplace type. Interestingly, all nonlinearities of the original equations give rise to the potential term in $\mathcal{P}$. An example of this construction is given in the case of a closed Friedmann–Lemaitre–Robertson–Walker universe. Eventually, the ADM equations are re-expressed as a coupled first-order system for the induced metric and the trace-free part of $K$. Such a system is written in a form which clarifies how a set of first-order differential operators and their inverses, jointly with spectral resolutions of operators of Laplace type, contribute to solving, at least in principle, the original ADM system.

The canonical formulation of general relativity relies on the assumption that space-time $(M,g)$ is topologically $\Sigma \times \mathbf{R}$ and can be foliated by a family of spacelike surfaces $\Sigma_t$, all diffeomorphic to the three-manifold $\Sigma$. The space-time metric $g$ is then locally cast in the form

$$g = -(N^2 - N_i N^i) dt \otimes dt + N_i (dx^i \otimes dt + dt \otimes dx^i) + h_{ij} dx^i \otimes dx^j, \tag{1}$$

where $N$ is the lapse function and $N^i$ are components of the shift vector of the foliation [1–3]. The induced metric $h_{ij}$ on $\Sigma_t$ and the associated extrinsic-curvature tensor $K_{ij}$ turn out to obey the first-order equations [1–3]

$$\frac{\partial h_{ij}}{\partial t} = -2NK_{ij} + N_{i|j} + N_{j|i}, \tag{2}$$

$$\frac{\partial K_{ij}}{\partial t} = -N_{|ij} + N \left[ {}^{(3)}R_{ij} + K_{ij}(\mathrm{tr}K) - 2K_{im}K_j^{\ m} \right] \\ + \left[ N^m K_{ij|m} + N^m_{\ |i} K_{jm} + N^m_{\ |j} K_{im} \right], \tag{3}$$

where the stroke denotes covariant differentiation with respect to the induced connection on $\Sigma_t$. Moreover, the constraint equations hold. For Einstein theory in vacuum they read

$$K_{il}^{\ \ |i} - (\mathrm{tr}K)_{|l} \approx 0, \tag{4}$$

$${}^{(3)}R - K_{ij}K^{ij} + (\mathrm{tr}K)^2 \approx 0, \tag{5}$$

where $\approx$ is the weak-equality symbol introduced by Dirac to denote equations which only hold on the constraint surface [3,4].

Equation (3) is quasi-linear in that it contains terms quadratic in the extrinsic-curvature tensor, i.e.

$$N \left( K_{ij}(\mathrm{tr}K) - 2K_{im}K_j^{\ m} \right).$$

We are now aiming to obtain from eqs. (2) and (3) another set of quasi-linear first-order equations. For this purpose, we contract Eq. (2) with $K^{ij}$ and Eq. (3) with $h^{ij}$. This leads to (hereafter $-\triangle \equiv -_{|i}^{\ \ |i}$ is the Laplacian on $\Sigma_t$)

$$K^{ij} \frac{\partial h_{ij}}{\partial t} = -2NK_{ij}K^{ij} + 2K^{ij}N_{i|j}, \tag{6}$$

$$h^{ij}\frac{\partial K_{ij}}{\partial t} = - \triangle N + N\left[{}^{(3)}R + (\text{tr}K)^2 - 2K_{im}K^{im}\right]$$
$$+ \left[N^m(\text{tr}K)_{|m} + 2N^m_{\ |i}K^i_{\ m}\right]. \tag{7}$$

It is now possible to obtain an evolution equation for $(\text{tr}K)$, because

$$\frac{\partial}{\partial t}(\text{tr}K) = \frac{\partial h^{ij}}{\partial t}K_{ij} + h^{ij}\frac{\partial K_{ij}}{\partial t}. \tag{8}$$

The second term on the right-hand side of Eq. (8) is given by Eq. (7), whereas the first term is obtained after using the identity

$$\frac{\partial}{\partial t}(h^{ij}h_{jl}) = \frac{\partial}{\partial t}\delta^i_{\ l} = 0, \tag{9}$$

which implies

$$\frac{\partial h^{ij}}{\partial t} = -h^{ip}h^{jl}\frac{\partial h_{pl}}{\partial t}, \tag{10}$$

and hence

$$\frac{\partial h^{ij}}{\partial t}K_{ij} = -\frac{\partial h_{ij}}{\partial t}K^{ij}. \tag{11}$$

By virtue of Eqs. (6)–(8) and (11), and imposing the constraint equation (5), we get

$$\frac{\partial}{\partial t}(\text{tr}K) = -\left(\triangle - K_{ij}K^{ij}\right)N + N^m(\text{tr}K)_{|m}, \tag{12$a$}$$

which can also be cast in the form (here $\widehat{\nabla}_m \equiv {}_{|m}$)

$$\left(\frac{\partial}{\partial t} - N^m\widehat{\nabla}_m\right)(\text{tr}K) = \left(- \triangle + K_{ij}K^{ij}\right)N. \tag{12$b$}$$

This is a first non-trivial result because it tells us that, given the operator of Laplace type [5]

$$\mathcal{P} \equiv - \triangle + K_{ij}K^{ij}, \tag{13}$$

the right-hand side of Eq. (12b) is determined by a discrete spectral resolution of $\mathcal{P}$. By this one means a complete orthonormal set of eigenfunctions $f^p_\lambda$ belonging to the eigenvalue $\lambda$, so that, *for each fixed value* of $t$, the lapse can be expanded in the form

$$N(\vec{x},t) = \sum_\lambda C_\lambda f^p_\lambda(\vec{x},t), \tag{14}$$

with Fourier coefficients $C_\lambda$ given by the scalar product

$$C_\lambda = \Bigl(f_\lambda^p, N\Bigr). \tag{15$a$}$$

This point is simple but non-trivial: for each fixed value of $t$, the restriction of the lapse to $\Sigma_t$ becomes a function on $\Sigma_t$ only, and hence the Fourier coefficients in (15a) read

$$C_\lambda = \int_{\Sigma_t} (f_\lambda^p)^* N \sqrt{h} d^3x \equiv C_{\lambda,t}, \tag{15$b$}$$

where the star denotes complex conjugation. In the application to the initial-value problem, one studies the lapse at different values of $t$, and hence it is better to write its expansion in the form (14), with the time parameter explicitly included. The integration measure for $C_\lambda$, however, remains the invariant integration measure on $\Sigma_t$.

The existence of discrete spectral resolutions of $\mathcal{P}$ is guaranteed if $\Sigma_t$ is a compact Riemannian manifold without boundary [5], which is what we assume hereafter. Thus, Eq. (12b) can be re-expressed in the form

$$\left(\frac{\partial}{\partial t} - N^m \widehat{\nabla}_m\right) \mathrm{tr}K(\vec{x},t) = \sum_\lambda \lambda C_{\lambda,t} f_\lambda^p(\vec{x},t). \tag{16}$$

On denoting by $L$ the operator

$$L \equiv \frac{\partial}{\partial t} - N^m \widehat{\nabla}_m, \tag{17}$$

and writing $G(\vec{x},t;\vec{y},t')$ for the Green function of $L$, Eq. (16) can be therefore solved for $(\mathrm{tr}K)$ in the form

$$\mathrm{tr}K(\vec{x},t) = \int_{\Sigma_{t'} \times \mathbf{R}} G(\vec{x},t;\vec{y},t') \sum_\lambda \lambda C_{\lambda,t'} f_\lambda^p(\vec{y},t') d\mu(\vec{y},t'), \tag{18}$$

where $d\mu(\vec{y},t')$ is the invariant integration measure on $\Sigma_{t'} \times \mathbf{R}$. Note that the eigenfunctions $f_\lambda^p$ are different at different times and hence, to compute them at all times, one has to integrate the equations of motion so that the metric and the extrinsic-curvature tensor are known at all times. Thus, Eq. (18) does not reduce the amount of entanglement of the original equations (2) and (3). A relevant particular case is obtained on considering

the canonical form of the foliation [6], for which the shift vector vanishes. The operator $L$ reduces then to $\frac{\partial}{\partial t}$, and its Green function can be chosen to be of the form

$$G_c(t,t') = \frac{1}{2}\Big[\theta(t-t') - \theta(t'-t)\Big], \tag{19}$$

where $\theta$ is the step function. Such a Green function equals $\frac{1}{2}$ for $t > t'$ and $-\frac{1}{2}$ for $t < t'$, and hence its "jump" at $t'$ equals 1, as it should be from the general theory. It is sometimes considered in a completely different branch of physics, i.e. the theory of solitons and nonlinear evolution equations.

A cosmological example shows how the right-hand side of Eq. (16) can be worked out explicitly in some cases. We are here concerned with closed Friedmann–Lemaitre–Robertson–Walker models, for which the three-manifold $\Sigma$ reduces to a three-sphere of radius $a$. It is indeed well known that the space of functions on the three-sphere can be decomposed by using the invariant subspaces corresponding to irreducible representations of $O(4)$, the orthogonal group in four dimensions. These invariant subspaces are spanned by the hyperspherical harmonics [7], i.e. generalizations to the three-sphere of the familiar spherical harmonics. The scalar hyperspherical harmonics $Q^{(n)}(\chi, \theta, \varphi)$ form a basis spanning the invariant subspace labeled by the integer $n \geq 1$ corresponding to the $n$-th scalar representation of $O(4)$. The label $(n)$ of $Q^{(n)}$ refers both to the order $n$ and to other labels denoting the different elements spanning the subspace. The number of elements $Q^{(n)}$ is determined by the dimension of the corresponding $O(4)$ representation. On using the general formula for $O(2k)$ representations, the dimension of the irreducible scalar representations is found to be $d_Q(n) = n^2$. The Laplacian on a unit three-sphere when acting on $Q^{(n)}$ gives

$$-\triangle\, Q^{(n)} = (n^2 - 1)Q^{(n)} \quad n = 1, 2, \dots \ . \tag{20}$$

Any arbitrary function $F$ on the three-sphere can be expanded in terms of these hyperspherical harmonics as they form a complete set:

$$F = \sum_{(n)} q_{(n)} Q^{(n)}, \tag{21}$$

where the sum $(n)$ runs over $n = 1$ to $\infty$ and over the $n^2$ elements in each invariant subspace, and the $q_{(n)}$ are constant coefficients. The lapse function on the three-sphere is therefore expanded according to (cf. (14))

$$N = \sum_{(n)} c_{(n)} Q^{(n)}, \tag{22}$$

and bearing in mind that $K_{ij}K^{ij} = 3$ on a unit three-sphere we find, on a three-sphere of radius $a$,

$$\mathcal{P}N = \sum_{(n)} c_{(n)} \lambda_n Q^{(n)}, \tag{23}$$

where (here $a = a(t)$)

$$\lambda_n = \frac{(n^2+2)}{a^2} \quad n = 1, 2, \ldots \ . \tag{24}$$

We have therefore found that all nonlinearities of the original equations give rise to the potential term in the operator $\mathcal{P}$ defined in Eq. (13), which is a positive-definite operator of Laplace type (unlike the Laplacian, which is bounded from below but has also a zero eigenvalue when $n = 1$).

The interest in the evolution equation for the trace of the extrinsic-curvature tensor is motivated by a careful analysis of the variational problem in general relativity. More precisely, the "cosmological action" is sometimes considered, which is the form of the action functional $I$ appropriate for the case when $(\mathrm{tr}K)$ and the conformal three-metric $\widetilde{h}_{ij} \equiv h^{-\frac{1}{3}}\ h_{ij}$ are fixed on the boundary. One then finds in $c = 1$ units [8]

$$I = \frac{1}{16\pi G} \int_M {}^{(4)}R\sqrt{-g}\ d^4x + \frac{1}{24\pi G} \int_{\partial M} (\mathrm{tr}K)\sqrt{h}\ d^3x, \tag{25}$$

and this formula finds important applications also to the quantum cosmology of a closed universe, giving rise to the $K$-representation of the wave function of the universe [9].

The program we have outlined in our letter shows an intriguing link between the Cauchy problem in general relativity on the one hand [10–13], and the spectral theory of elliptic operators of Laplace type on the other hand [5], which is obtained by exploiting the nonlinear form of the ADM equations and the nonpolynomial form of the Hamiltonian constraint (5). On denoting by $\sigma_{ij}$ the trace-free part of the extrinsic-curvature tensor:

$$\sigma_{ij} \equiv K_{ij} - \frac{1}{3} h_{ij} (\mathrm{tr}K) \tag{26}$$

we therefore obtain a set of equations, equivalent to (2) and (3), in the form (here $L^{-1}$ denotes the inverse of the operator $L$ defined in (17))

$$\mathrm{tr}K = L^{-1}\ \mathcal{P}N, \tag{27}$$

$$\frac{\partial h_{ij}}{\partial t} = -\frac{2}{3} N h_{ij} (\mathrm{tr}K) - 2N\sigma_{ij} + 2N_{(i|j)}, \tag{28}$$

$$\frac{\partial \sigma_{ij}}{\partial t} = \Omega_{ij} N - \frac{2}{3} N_{(i|j)} (\mathrm{tr} K) + N^m \widehat{\nabla}_m \sigma_{ij} + 2 N^m_{(|i} \Big[ \sigma_{j)m} + \frac{1}{3} h_{j)m} (\mathrm{tr} K) \Big], \tag{29}$$

where the operator of Laplace type $\Omega_{ij}$ is given by

$$\Omega_{ij} \equiv - \widehat{\nabla}_j \widehat{\nabla}_i - \frac{1}{3} h_{ij} \mathcal{P} + \frac{1}{3} \sigma_{ij} (\mathrm{tr} K) + \frac{1}{3} h_{ij} (\mathrm{tr} K)^2 + {}^{(3)}R_{ij} - 2 \sigma_{im} \sigma_j^{\ m}. \tag{30}$$

It now appears desirable to understand to which extent the differential-operator viewpoint plays a role in solving the coupled system (27)–(30). For this purpose, we consider a relevant particular case, i.e. the canonical foliation studied in Ref. [6], for which the shift vector vanishes. The operator $L$ reduces then to $\frac{\partial}{\partial t}$ and, on defining the operators

$$Q \equiv \frac{\partial}{\partial t} + \frac{2}{3} N \mathrm{tr} K = L + \frac{2}{3} N L^{-1} \mathcal{P} N, \tag{31}$$

$$S \equiv \frac{\partial}{\partial t} - \frac{N}{3} \mathrm{tr} K = L - \frac{N}{3} L^{-1} \mathcal{P} N, \tag{32}$$

$$A_{ij} \equiv - \widehat{\nabla}_j \widehat{\nabla}_i + {}^{(3)}R_{ij}, \tag{33}$$

one finds the equations

$$\left[ S + \frac{2}{3} (L^{-1} \mathcal{P} N)^2 Q^{-1} N \right] \sigma_{ij} - \frac{2}{3} (Q^{-1} N \sigma_{ij}) \mathcal{P} N + 2 N \sigma_{im} \sigma_j^{\ m} = A_{ij} N, \tag{34}$$

$$h_{ij} = -2 Q^{-1} N \sigma_{ij}, \tag{35}$$

supplemented, of course, by Eq. (27). In other words, once Eq. (34) is solved for the trace-free part of the extrinsic-curvature tensor of $\Sigma_t$, the induced metric on $\Sigma_t$ is obtained from Eq. (35). This form of the coupled first-order ADM system of equations has possibly the merit of stressing the need to invert the operators $L, Q$ and $S$ to find a solution for given initial conditions. Approximate solutions, to the desired accuracy, will correspond to the construction of approximate inverse operators $L^{-1}, Q^{-1}$ and $S^{-1}$. For this purpose, it can be useful to re-express Eq. (34) in the form

$$\left[ I + \frac{2}{3} S^{-1} (L^{-1} \mathcal{P} N)^2 Q^{-1} N \right] \sigma_{ij} - \frac{2}{3} S^{-1} \left[ (Q^{-1} N \sigma_{ij}) \mathcal{P} N \right] - S^{-1} (A_{ij} N) = -2 S^{-1} \left( N \sigma_{im} \sigma_j^{\ m} \right). \tag{34'}$$

The action of the inverse operator $L^{-1}$ on $\mathcal{P}N$ is already given by Eq. (18), but the inverses $Q^{-1}$ and $S^{-1}$ make it necessary to develop an algorithm for their exact or approximate evaluation. More precisely, one has

$$Q = L\left(I + \frac{2}{3}L^{-1}NL^{-1}\mathcal{P}N\right), \tag{31$'$}$$

$$S = L\left(I - \frac{1}{3}L^{-1}NL^{-1}\mathcal{P}N\right), \tag{32$'$}$$

and hence

$$Q^{-1} = \left(I + \frac{2}{3}L^{-1}NL^{-1}\mathcal{P}N\right)^{-1} L^{-1}, \tag{36}$$

$$S^{-1} = \left(I - \frac{1}{3}L^{-1}NL^{-1}\mathcal{P}N\right)^{-1} L^{-1}, \tag{37}$$

by virtue of the operator identity $(AB)^{-1} = B^{-1}A^{-1}$. From this point of view, two levels of approximation seem to emerge:

(i) The degree of accuracy in the evaluation of $Q^{-1}$ and $S^{-1}$. It is clear from Eqs. (36) and (37) that this reduces to finding

$$\left(I + \rho L^{-1}NL^{-1}\mathcal{P}N\right)^{-1}$$

where $\rho = \frac{2}{3}$ or $-\frac{1}{3}$. A series expansion will be, in general, only of formal value. However, on using the norms defined in section 7.4 of Ref. [14], which relies on well known properties of Sobolev spaces, if the operator

$$T_\rho \equiv \rho L^{-1}NL^{-1}\mathcal{P}N \tag{38}$$

is a bounded operator on a Banach space $X$ with norm $\|T_\rho\| < 1$, the inverse of $I + T\rho$ exists and the series $\sum_{n=0}^{\infty}(-1)^n(T_\rho)^n$ converges uniformly to $(I + T_\rho)^{-1}$ with respect to the norm on the set of bounded maps from $X$ into $X$, and one can write

$$(I + T_\rho)^{-1} = \sum_{n=0}^{\infty}(-1)^n(T_\rho)^n. \tag{39}$$

Equation (39) can be therefore used if $L^{-1}$ is such that $T_\rho$ is a bounded operator with $\|T_\rho\| < 1$.

(ii) The way in which the non-linear term $\sigma_{im}\sigma_j^{\ m}$ is dealt with. For example, one may regard the right-hand side of Eq. (34') as the "known term", and hence consider the equation

$$\sigma_{ij} dx^i \otimes dx^j - \frac{2}{3} F^{-1} S^{-1} \left[ (Q^{-1} N \sigma_{ij}) \mathcal{P} N \right] dx^i \otimes dx^j$$
$$- F^{-1} S^{-1} (A_{ij} N) dx^i \otimes dx^j = -2 F^{-1} S^{-1} \left( N \sigma_{im} \sigma_j^{\ m} \right) dx^i \otimes dx^j, \tag{40}$$

where $F^{-1}$ is the inverse of the operator

$$F \equiv I + \frac{2}{3} S^{-1} (L^{-1} \mathcal{P} N)^2 Q^{-1} N. \tag{41}$$

Equation (40) is an integral equation for $\sigma_{ij}$, for which a perturbation approach might be useful if one takes

$$-2 F^{-1} S^{-1} \left( N \sigma_{im} \sigma_j^{\ m} \right) dx^i \otimes dx^j$$

as the known term mentioned before. Note that we have resorted to the use of the tensor product $dx^i \otimes dx^j$ because in the resulting equation (40) one has a well defined integration of a "symmetric two-form" over the space-time manifold, here taken to be diffeomorphic to $\Sigma_t \times \mathbf{R}$. Strictly, also Eq. (35) should be written as

$$h_{ij} dx^i \otimes dx^j = -2 \Big( Q^{-1} N \sigma_{ij} \Big) dx^i \otimes dx^j, \tag{35$'$}$$

and in all such equations only the symmetric part of the tensor product survives, because both $h_{ij}$ and $\sigma_{ij}$ are symmetric rank-two tensor fields.

Note once more that it would be wrong to regard Eq. (27) as independent of the solution of Eqs. (28) and (29), because $L^{-1}\ \mathcal{P} N$ is only known at all times when the time evolution of $h_{ij}$ and $K_{ij}$ has been determined for given initial conditions. Our equations (35')–(37) and (40), although not obviously more powerful than previous schemes, prepare the ground for an operator approach to the ADM equations for general relativity, and hence might contribute to the understanding of structural properties of general relativity. In particular, our way of writing the coupled system of non-linear evolution equations might lead to a better understanding of the interplay between elliptic operators on the spacelike surfaces $\Sigma_t$ and hyperbolic equations on the space-time manifold. As far as we can see, this expectation is supported by the closed Friedmann–Lemaitre–Robertson–Walker model discussed before (where $\sigma_{ij}$ vanishes), and further examples of cosmological interest might be found, e.g. Bianchi IX models describing a closed but anisotropic universe. For open universes or asymptotically flat space-times, however, we are not aware of theorems that

make it possible to expand the lapse as in (14). In such cases, the counterpart of the elliptic theory on $\Sigma_t$ advocated in our paper is therefore another open problem.

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